CONSTRUCTION ROUTES IN THE SOLUTION OF COMPASS AND STRAIGHTEDGE CONSTRUCTION PROBLEMS: GEOMETRIC REASONING OF FIRST YEAR MATHEMATICS EDUCATION STUDENTS

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In this paper we share our initial analysis of the construction routes of a sample of 23 first year mathematics education students who solved a compass and straightedge construction problem as an assessment task. Four construction routes were identified which provide a window on their geometric reasoning about the rhombus they had to construct. We describe the apparent theoretical geometric reasoning and spatio-graphic reasoning of each construction route and reflect on possible reasons for unsuccessful constructions. These initial findings are important for teachers and teacher educators who want to use constructions as a tool for the development of geometric reasoning.

INTRODUCTION

Mathematics teacher education in South Africa, as in other developing countries has to take note of tendencies in developed countries to shift towards the use of dynamic computer based learning in schools. Yet, in South Africa the reality is that the majority of schools do not have access to the necessary technology and the majority of teachers do not have the knowledge to use computer based artifacts for teaching. With the eye on the future mathematics teacher education nevertheless has to provide access to such technology and develop the reasoning to accompany learning and teaching with technology. At the same time it has to provide alternative non-technological tools for teaching which can provide access to computer age reasoning. We argue that the compass and straightedge is such a tool.

The South African school mathematics curriculum includes compass and straightedge constructions in Grades 7 to 9, but judging from examples in the CAPS document actual construction tasks tend to be procedural and limited to a list of basic constructions, such as the construction of a line perpendicular to another, halving a line segment and duplicating and halving angles. Evident from textbooks, the basic constructions are taught as an end in themselves and teachers are not likely to utilize them in geometric problem-solving or to develop geometric reasoning. In this paper we analyse the construction routes of a sample of 23 first year mathematics education students who constructed a rhombus from limited theoretical information. The construction routes provide information about their geometric reasoning.

CONSTRUCTIONS AND PROOF REASONING

Martin (1998, p. 2) explains the relationship between theorems and proofs and constructions as follows:

"In general, a *theorem* is a statement that has a proof based on a give set of postulates and previously proved theorems. A *proof* is a convincing argument. A problem in Euclid asks that some new geometric entity be created from a given set. We call a solution to such a problem a *construction*. This construction is in itself a theorem, requiring a proof and having the form of a recipe: If you do this, this, and this, then you will get that. Such a mathematical recipe is called an algorithm. So a construction is a special type of theorem that is also an algorithm. (We hesitatingly offer the analogy: *Problem*: Make a pudding; *Construction*: Recipe; *Proof*: Eating)"

Martin's baking analogy suggests that the proof is only the spatio-graphical, perceptual object that is the end product of the construction. Laborde (2005, p. 174) is more explicit about the demands posed by construction problems, namely to produce mappings between the theoretical and spatio-graphic domains. With a compass and straightedge construction the spatio-graphic properties provide the visual verification of the correctness of the constructed figure, while the theoretical properties guide the construction process. In the construction problems we posed in our first year geometry course for mathematics education students, an algorithm has to be developed based on the theoretical-geometric properties of the end product and the affordances and constraints of the compass and straightedge as tools. The algorithm and the spatio-graphic object together serve to provide the convincing argument. According to Laborde (2005, p. 160) a key aspect of geometry learning and reasoning is to distinguish between incidental and necessary spatio-graphical properties of geometric objects. This is exactly the opportunity afforded by Euclidean constructions. Whether done with paper, pencil and compass and straightedge, or with dynamic tools, Euclidean constructions are based solely on distance-relationships between points, and reasoning has to start with the construction of a basic length or distance (the radius of the initial circle). All points that are used in the subsequent algorithm have to be determined as intersections of arcs, lines or segments, or endpoints of segments. Points that are not created by such intersections or segmentations have arbitrary or incidental position. In South-African classrooms learners are very seldom required to draw or construct the geometrical object of a problem and textbooks provide diagrams for all problems, with the result that learners often find it difficult to discern the necessary or incidental properties of points and segments. Our observations about students' strategies to solve riders is that they tend to fill in on a given diagram all immediately available measurements or congruencies and then hope for a solution to "jump in the eye". They are hard pressed to reason about the primacy relationships between parts of the figure. We concur with Laborde (2005) that it is detrimental to geometric reasoning when theoretical properties are stressed almost to the exclusion of spatio-graphical properties in teaching.

CONSTRUCTIONS AND THE VAN HIELE LEVELS OF GEOMETRIC REASONING

The Van Hiele (1959) theory of geometric reasoning holds that such reasoning develops from gestalt-like visual reasoning, described as recognition at Level 1, to reasoning based on the analysis of the properties of geometric objects at Level 2, and further to reasoning with the ordered relationships between properties of objects at Level 3. At Level 4 learners are able to develop longer sequences of deduction and begin to understand theorems and proofs. The reasoning levels are hierarchical and language or verbal reasoning plays a significant role in the development through the levels. Van Hiele stressed the importance of active tactile involvement of learners in order to develop their reasoning from visual to analytic. In the same vein De Villiers claims that the transition from Level 1 to Level 2 "involves a transition from enactive-iconic handling of concepts to a more symbolic one." (De Villiers, 2010, p. 2)

While we were aware that our students were unlikely to be at Level 3, since geometry was not compulsory in their high school curriculum, we aimed to challenge our students to reason with the logical relationships between properties of figures. Such reasoning is described as Van Hiele Level 3 reasoning and indicated by the objects of their reasoning such as "noticing and formulating logical relationships between properties, for example that equal opposite sides implies that the sides are parallel." (De Villiers, 2010, p. 3) Until dynamic geometry programs became available research about Van Hiele thought levels was mostly based on the verbal contributions of participants. For example, Burger and Shaugnessy's (1986) descriptors of Van Hiele level reasoning is given exclusively in terms what participants say, and makes no mention of actions and their implications for geometric reasoning. Van Hiele research conducted in contexts of geometry software describes the dragging and construction actions of participants as they investigate and solve problems. The body of evidence is growing that the opportunities for goal directed action on geometric objects is an important aspect in the development of geometric reasoning. (De Villiers, 2010; Idris, 2009) Yet, we are not aware of research that analyse geometric reasoning during compass and straightedge construction in terms of Van Hiele levels. Our analysis of the task demands of problem-solving by construction is that at least Level 3 reasoning is required.

VISUAL AND THEORETICAL ASPECTS OF CONSTRUCTIONS

In the context of goal-directed action on dynamic geometric objects, the issues of the role of visualisation and the influence of the spatio-graphic diagram in relation to reasoning are given new priority. Early proponents of the importance of spatio-graphic properties like Fischbein (1993) and Mariotti (1995) describe geometric reasoning as conceptual-figural, and highlight pervasive conflicts between the two modes of reasoning. More recently Laborde (2005) reformulated the theoretical and visual aspects of geometric reasoning in terms of the referents rather than mental actions. Fischbein and Mariotti refer to geometric reasoning in non-dynamic contexts, while Laborde takes her thesis from research in dynamic contexts. Laborde (2005, p. 161) distinguishes between two domains, the *theoretical domain* (T) comprising of geometrical objects and relations; and the *spatio-graphic domain* (SG) – that of diagrams on paper or on the computer screen, or, importantly "movement produced by a linkage point of a machine." The theoretical and spatio-graphic domains are not independent and geometric reasoning requires repeated moves between the domains. According to Laborde geometry teaching should aim at integrating the T and SG domains. We argue that a pair of compasses is a machine with a linkage point, and as such included in the spatio-graphic domain, while constructions with these tools require reference to geometric relationships between properties of the object to be constructed.

As predicted by Van Hiele theory, the visual aspects of constructions are most influential for novice constructors. In the one study we could find of reasoning during compass constructions, Tapan and Arslan (2009) analysed pre-service teachers' compass constructions and accompanying justifications and concluded that visual and naïve empirical reasoning were the norm. The participants in their study struggled to organize constructions into theoretically sound algorithms.

THEORETICAL FRAMEWORK

We framed our research according to Laborde's (2005) distinction between the use of theoretical and spatio-graphic properties in geometric constructions - in particular, her thesis that the meaning of a geometrical activity resides in the weights of these properties as they interplay during construction. Analysis of the referents in a construction algorithm points to whether a student assigns theoretical or spatio-graphic meaning to the task. In other words, whether a construction is judged as successful based on mainly theoretical or visually observable spatio-graphic properties. This framework is an initial attempt to gain information about students' geometric reasoning from their constructions. We analyse the theoretical and spatio-graphic properties used in each construction route, and infer the meaning assigned to the construction. In our analysis the meaning emerges from an analysis of the properties that fully determine the outcome of the construction.

For example, if the construction marks indicate that side lengths were measured and constructed when they would have been determined by the intersection of rays, we infer that the spatio-graphic referent of equal side lengths guided the student, rather than the theoretical sufficiency of the preceding steps in the construction algorithm.

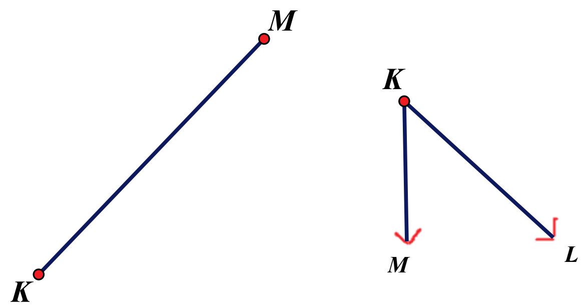
RESEARCH QUESTION AND METHOD

We aim to get initial answers the following questions:

1) What construction routes or algorithms are used when first year mathematics education students solve a construction problem?
2) What are the theoretical and spatio-graphic reasoning evident from the construction routes?
3) What meaning (theoretical or spatio-graphic) is assigned to the construction as evident from the concluding steps in the construction?

The construction problem

Construct a rhombus. KM is a diagonal of the rhombus and MKL is an angle. Use only compasses and a straightedge.



The students' constructions were categorized as successful or unsuccessful; and each category was further categorized in terms of the construction route. This allowed us to compare inferred reasoning processes and to develop hypotheses about factors that constrained the solution of the problem.

The construction problem draws on reasoning with the properties of a rhombus that are dependent on the diagonal. In order to solve the problem, students have to image the end product – a rhombus with diagonal MK and an angle MKL. The position of angle MKL in the rhombus requires an interpretation of the labels and knowledge of the conventions of labeling. The leg KM of the angle is also the diagonal KM, but while the length of the diagonal is determined, the lengths of the legs of the angle are not determined. Once the labeling conventions are sorted out, several solution routes are available based on the properties of the rhombus that are dependent on the diagonal.

RESULTS

Four major successful construction routes were identified.

Construction route 1 (n=2)

Construct KM. At K and M construct angles congruent to angle MKL to form isosceles triangle KML. Complete the rhombus by constructing KN and MN congruent to KL.

Spatio-graphic reasoning: A rhombus has four congruent sides; a diagonal of a rhombus bisects the rhombus into two congruent isosceles triangles.

Theoretical reasoning: The given angle on the diagonal is one of the base angles of an isosceles triangle on the diagonal. The size of the given angle and the length of the diagonal determine the length of the sides of the rhombus.

Meaning of the construction: Spatio-graphic: Find the length of the sides of the rhombus and construct all four sides. (The final step is the measurement and the construction of the remaining two isosceles sides.)

Construction route 2 (n=8)

Construct line segment KM. At K, construct angle LKN equal to double angle MKL. At M, construct angle LMN equal to double angle MKL. The intersections of the arms of angle LKN and angle LMN completes the rhombus.

Spatio-graphic reasoning: The diagonals of a rhombus are symmetry lines. (The second diagonal is often drawn but no properties of the intersection of the diagonals are indicated with markings or tested by construction).

Theoretical reasoning: If the given diagonal of a rhombus is a symmetry line, then the angles at the same vertex on either side of the diagonal are congruent. With this construction route the rhombus is fully determined once angles LKN and LMN are constructed.

Meaning of the construction: Theoretical: Congruent sides are the result of equal opposite angles bisected by a diagonal.

Construction route 3 (n= 1)

Construct Segment KM and its perpendicular bisector. At K construct angle MKL congruent to the given angle. The intersection of leg KL and the perpendicular bisector of KM defines the side length of the rhombus. Connect M with L. Complete the rhombus by constructing sides KN and MN congruent to KL. (A variation on this route is to construct two adjacent angles, each congruent to MKL, at K and hence obtain two intersections with the perpendicular bisector of KM).

Theoretical reasoning: If the second diagonal bisects the given diagonal perpendicularly then the given angle determines the position of the vertex formed by the intersection of the second diagonal and the leg of the angle which is not the given diagonal. The rhombus is fully determined when the right triangle between the diagonals is constructed.

Spatio-graphic reasoning: The second diagonal is bisected (perpendicularly) by the given diagonal. Markings indicate the congruent sections of the second diagonal, but the congruence is not constructed or tested. Opposite sides are marked parallel purely on visual grounds. Parallel properties are not verified.

Meaning of the construction: Spatio-graphic: Find the length of a side of the rhombus and construct all four sides. (The bisection of the second diagonal is not used to construct the fourth vertex, but the constructed side length is copied and constructed explicitly three more times.)

Construction route 4 (n=6)

Construct line segment KM and its perpendicular bisector. At K construct angle MKL congruent to the given angle, and at M construct the alternate angle KMN congruent to the given angle. The intersections of the angles with the perpendicular bisector of KM define the side lengths of the rhombus. Complete the rhombus connecting L with M and K with N.

Theoretical reasoning: At least one diagonal of a rhombus bisects the other diagonal perpendicularly; opposite sides of a rhombus are parallel, alternate angles formed by the diagonal and the sides of a rhombus are congruent.

Spatio-graphic properties: Sides of the rhombus are congruent and parallel.

Meaning of the construction: Theoretical: The length of the sides need not be explicitly determined and constructed. Congruent sides are the result of one pair of opposite and parallel sides intersecting with a diagonal of the rhombus.

Unsuccessful construction routes

Analysing the construction routes of unsuccessful students brings into focus the influence of the spatio-graphic properties of the object during the construction process. The students who failed to construct a spatio-graphically correct rhombus, but who had theoretically correct construction routes failed to integrate the labeling of the given angle and diagonal segment with the spatio-graphical object. Their constructions utilize the given angle MKL as an internal angle of the rhombus, and the actual lengths of the legs of the given angle as side lengths of their rhombi. We interpret this as indicative of visual reasoning at Van Hiele Level 1 – within a holistic image of a rhombus, angles are at the corners of the rhombus, and the legs of such angles are sides of the rhombus. Yet these students (5 in total) proceeded to reason successfully with the property that the diagonal of a rhombus bisects opposite angles to construct a spatio-graphically correct rhombus.

Only one student in the sample was completely unsuccessful. The student failed both logically and spatio-graphically to construct a rhombus from the given parts. The student's construction suggests that he held captive by his visual perception of a rhombus. He drew the diagonal KM in a slanted orientation to the page, then drew a segment KN parallel to the edge of the page and another segment PM visually parallel to KN and the edge of the page. He ended with an attempt to fit the given angle MKL on the diagonal, but as it did not fit the space between the diagonal and the drawn segments, he ended with a drawing (rather than a construction) of a parallelogram. Note that the student was able to copy-and-construct the given angle onto the diagonal, but he did not understand the relationship between this angle and the sides of the rhombus.

DISCUSSION

The distinction between the use of geometric properties and spatio-graphic properties to solve the construction problem is for us a point of contention. Without access to students' verbal justification of each construction step one cannot be sure whether the step is based on spatio-graphical or theoretical information. However, our inference of the meaning of the construction based on the final construction step holds potential for interpreting the overall meaning of the construction. If the spatio-graphic success of the construction was seen as a necessary consequence of relationships between the angle and diagonal properties of a rhombus, all four side lengths did not have to be explicitly constructed. In construction routes 1 and 3, we inferred the meaning of the construction to be "find the side lengths of the rhombus", since the construction press points and arcs indicate that the constructions were completed by measuring the length of the side obtained by constructing the angle and diagonal in the correct relationship, and purposefully constructing the remaining sides. We infer that the conclusion of the rhombus in this way indicate more weight to the visually observable, holistic, spatio-graphic properties of a rhombus. Indeed, in comparison with constructions routes 2 and 4, the reasoning seems to be more on the level of reasoning with visually observable properties of the rhombus (Van Hiele Level 1 or 2) than reasoning with relationships between the properties (van Hiele Level 3).

In construction routes 2 and 4, the rhombus was completed without purposefully constructing the side lengths. The students seem to have trusted that the property of equal and/or parallel sides will follow as a result of the relationship between the diagonal and the given angle. This would be indicative of Van Hiele Level 3 reasoning. There are no press points or construction arcs to indicate measurement and construction – the relevant rays were simply extended to where they intersect the second diagonal, and the intersection points labeled. We concur with Laborde (2005) that the conclusion and verification of the constructions are based on both the theoretical and spatio-graphic properties of the constructed object and we need to interview students about their constructions to better understand the moves between theoretical and spatio-graphic information.

CONCLUSION

We found evidence in the construction routes of the students which allows us to infer Van Hiele reasoning levels. Succesful students seem to have reasoned with the relationships between the properties of the rhombus, although they showed differences in what they seem to have construed as the meaning of the construction. Constraining factors seem to emerge from both spatio-graphic and theoretical domains as well. Less successful constructions were marred according to spatio-graphical criteria, despite the theoretical correctness of the construction routes. These students also reasoned with the relationships between the properties of a rhombus, but they misappropriated spatio-graphical information like labeling on the given parts in relation to the completed rhombus. While we need further research that provides access to students verbal justifications for construction steps, we are encouraged by the geometric reasoning that is apparent in the construction routes we identified.

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